ON THE APPROXIMATION PROPERTIES OF CESÀRO MEANS OF NEGATIVE ORDER OF VILENKIN-FOURIER SERIES

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Abstract. In this paper we establish approximation properties of Cesàro $(C,-\alpha)$ means with $\alpha \in (0,1)$ of Vilenkin-Fourier series.This result allows one to obtain the condition which is sufficient for the convergence of the means $\sigma_n^{-\alpha}(f,x)$ to $f(x)$ in the L^p-metric.

1. Introduction

Let N_+ denote the set of positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not lass then 2. Denote by $Z_{m_k} := \{0, 1, ..., m_k-1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j} with the product of the discrete topologies of Z_{mj}'s.

The direct product of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \qquad (j \in Z_{m_k}).$$

Is the Haar measure on G_m with $\mu(G_m) = 1$. If the sequence m is bounded, then G_m is called a bounded Vilenkin group. The elements of G_m can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, $(x_j \in Z_{m_j})$. The group operation $+$ in G_m is given by

$$x + y = ((x_0 + y_0) \, mod m_0, ..., (x_k + y_k) \, mod m_k, ...),$$

where $x := (x_0, ..., x_k, ...)$ and $y := (y_0, ..., y_k, ...) \in G_m$.

The inverse of $+$ will be denoted by $-$.It is easy to give a base for the neighborhoods of G_m :

$$I_0(x) := G_m$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\}.$$

for $x \in G_m, n \in N$ define $I_n = I_n(0)$ for $n \in N_+$. Set $e_n := (0, ..., 0, 1, 0, ...) \in G_m$ the n th coordinate of which is 1 and the rest are zeros $(n \in N)$.

If we define the so-called generalized number system based on m in the following way: $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in N)$, then every $n \in N$ can be uniquely expressed as $n=\sum\limits_{j=0}^{\infty}n_jM_j$, where $n_j\in\ Z_{mj}$ $(j\in N_+)$ and only a finite number of n_j's differ from zero. We use the following notation. Let $(for\ n>0)$ $|n|:=max\{k\in N:n_k\neq 0\}$ (that is , $M_{|n|}\leq n<M_{|n|+1}$). Next, we introduce of G_m an orthonormal system which is called Vilenkin system. At first define the complex valued functions $r_k\left(x\right):G_m\rightarrow C$. the generalized Rademacher functions in this way

$$r_k(x):=\exp\left(\frac{2\pi ix_k}{m_k}\right),\qquad \left(i^2=-1,\ x\in G_m,\ k\ \in\ N\right).$$

New define the Vilenkin system $\psi:=(\psi_n:n\in N)$ on G_m as follows.

$$\psi_n\left(x\right):=\prod_{k=0}^{\infty}r_k^{n_k}\left(x\right),\qquad\left(n\ \epsilon\ N\right).$$

Specifically, we call the system the Walsh-Paley on if $m=2$. The Vilenkin system is orthonormal and complete in $L^1\left(G_m\right)$. Now, introduce analogues of the usual definitions of the Fourier analysis.If $f\in L^1\left(G_m\right)$ we can establish the fallowing definitions in the usual way:

Fourier coefficients:

$$\widehat{f}\left(k\right):=\int\limits_{G_m}f\overline{\psi}d\mu,\qquad\left(k\ \in N\right),$$

partial sums:

$$S_nf:=\sum_{k=0}^{n-1}\widehat{f}\left(k\right)\psi_k,\qquad\left(n\ \in\ N_+\ ,\ S_0f:=0\right),$$

Fejér means:

$$\sigma_nf:=\frac{1}{n}\sum_{k=1}^{n}S_kf\ ,\qquad\left(n\ \epsilon\ N_+\right).$$

Dirichled kernels:

$$D_n:=\sum_{k=0}^{n-1}\psi_k,\qquad\left(n\ \in N_+\right).$$

Fejér kernels:

$$K_n(x):=\frac{1}{n}\sum_{k=1}^{n}D_k\left(x\right).$$

Recall that

$$D_{M_n}\left(x\right)=\left\{\begin{array}{ll}M_n, & \text{if } x\in I_n,\\ 0, & \text{if } \ x\notin I_n.\end{array}\right.\tag{1}$$

It is well Known that

$$\sigma_n f(x) = \int\limits_{G_m} f(t) K_n(x-t) \, d\mu(t).$$

The (c,α) means of the Vilenkin-Fourier series are defined as

$$\sigma_n^{-\alpha}(f,x) = \frac{1}{A_n^{-\alpha}} \sum_{k=0}^{n} A_{n-k}^{-\alpha} \widehat{f}(k) \psi_k(x).$$

where

$$A_0^{\alpha} = 1, \qquad A_n^{\alpha} = \frac{(\alpha+1)\dots(\alpha+n)}{n!}.$$

It is well Known that [11]

$$A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}. \tag{2}$$

$$A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}. \tag{3}$$

$$A_n^{\alpha} \sim n^{\alpha}. \tag{4}$$

The norm (or quasinorm) of the space $L^p(G_m)$ is defined by

$$\|f\|_p := \left(\int\limits_{G} |f(x)|^p \, d\mu(x) \right)^{1/p}, \qquad (0 < p < \infty)$$

If $f \in L^p(G_m)$, then

$$\omega\left(\frac{1}{M_n}, f\right)_p = \sup_{h \in I_n} \|f(\cdot + h) - f(\cdot)\|_p.$$

The problems of summability of Cesàro means of positive order for walsh-Fourier series were studied in [7].

Tevzadze [9] has studied the uniform convergence of Cesàro means of negative order for walsh-Fourier series.In particular,in terma of moduli of continiuty and variation of function $f \in C_\omega([0,1])$ he has proved the criterion for the uniform summability by the Cesàro method of negative order of Fourier series with respect to the Walsh system.

In [4],[5] U.Goginava proved conditions sufficient for the convergence of Cesàro means of negative order for walsh-Fourier series in space $L^p([0,1])$,$1 \le p \le \infty$.

In [4],[5],[9] the results were estabilished without estimation of approximation.

In his monography [10] Zhizhinashvili investigated the behavior of Cesàro method of negative order for trigonometricFourier series in detail.U.Goginava studied analogical question in case fo the Walsh system.

Theorem 1. *Let f belong to $L^p([0,1])$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then for any $2^k\le n<2^{k+1}$ $(k,n\in N)$ the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p\le c(p,\alpha)\left\{2^{k\alpha}\omega\left(1/2^{k-1},f\right)_p+\sum_{r=0}^{k-2}2^{r-k}\omega\left(1/2^r,f\right)_p\right\}$$

holds true.

In this paper we study analogical questions in case Vilenkin system.

Theorem 2. *Let f belong to $L^p(G_m)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then for any $M_k\le n<M_{k+1}$ $(k,n\in N)$ the inequality*

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p\le c(p,\alpha)\left\{M_k^{\alpha}\omega\left(1/M_{k-1},f\right)_p+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p\right\}$$

holds true.

Corollary 1. *Let f belong to $L^p(G_m)$ for some $p\in[0,+\infty]$ and let*

$$M_k^{\alpha}\omega\left(1/M_{k-1},f\right)_p\to0,\qquad as\qquad k\to\infty,\qquad(0<\alpha<1).$$

Then

$$\left\|\sigma_n^{-\alpha}(f)-f\right\|_p\to0\qquad as\qquad n\to\infty.$$

2. AUXILIARY RESULTS

Lemma 1. *Let $\alpha_1,...,\alpha_n$ be real numbers.Then*

$$\frac{1}{n}\int_G\left|\sum_{k=1}^n\alpha_kD_k(x)\right|d\mu(x)\le\frac{c}{\sqrt{n}}\left(\sum_{k=1}^n\alpha_k^2\right)^{1/2}.$$

where c is an absolute constant.

Lemma 2. *Let $f\in L^p(G_m)$ for some $p\in[1,\infty]$. Then for every $\alpha\in(0,1)$ the following estimations holds*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=0}^{M_{k-1}-1}A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$\le c(p,\alpha)\sum_{r=0}^{k-1}\frac{M_r}{M_k}\omega\left(1/M_k,f\right)_p,$$

where $M_k \leq n \leq M_{k+1}$.

Proof of Lemma 2. Applying Abel's transformation,from (3) we get

$$\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=0}^{M_{k-1}-1}A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p \tag{5}$$

$$=\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=1}^{M_{k-1}}A_{n-v-1}^{-\alpha}\psi_{v-1}(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$\leq\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=1}^{M_{k-1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\left[f(.\oplus u)-f(.)\right]d\mu(u)\right\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}A_{n-M_{k-1}-1}^{-\alpha}D_{M_{k-1}}(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p=I_1+I_2.$$

From the generalized Minkowski inequality, and by (1) and (4) we obtain

$$I_2\leq c(\alpha)M_{k-1}\int_0^{1/M_{k-1}}\left\|\left[f(\cdot\oplus u)-f(\cdot)\right]\right\|_p d\mu(u) \tag{6}$$

$$=O\left(\omega\left(1/M_{k-1},f\right)\right)_p,$$

$$I_1\leq\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p \tag{7}$$

$$\leq\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\left[f(\cdot\oplus u)-S_{M_r}(\cdot\oplus u,f)\right]d\mu(u)\right\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\left[S_{M_r}(\cdot\oplus u,f)-S_{M_r}(\cdot,f)\right]d\mu(u)\right\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\left\|\int_{G_m}\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v(u)\left[S_{M_r}(\cdot,f)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$=I_{11}+I_{12}+I_{13}.$$

Using Lemma 1 for I_{11}we can write

$$I_{11} \tag{8}$$

$$\leq \frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\int_{G_m}\left|\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v\left(u\right)\right|\left\|f\left(\cdot\oplus u\right)-S_{M_r}\left(\cdot\oplus u,f\right)\right\|_p d\mu\left(u\right)$$

$$\leq \frac{1}{A_n^{-\alpha}}\sum_{r=0}^{k-2}\omega\left(1/M_r,f\right)_p\int_{G_m}\left|\sum_{v=M_r}^{M_{r+1}-1}A_{n-v-1}^{-\alpha-1}D_v\left(u\right)\right|d\mu\left(u\right)$$

$$\leq c\left(\alpha\right)n^{\alpha}\sum_{r=0}^{k-2}\omega\left(1/M_r,f\right)_p\sqrt{M_{r+1}}\left(\sum_{v=M_r}^{M_{r+1}-1}\left(n-v-1\right)^{-2\alpha-2}\right)^{1/2}$$

$$\leq c\left(\alpha\right)n^{\alpha}\sum_{r=0}^{k-2}\omega\left(1/M_r,f\right)_p\sqrt{M_{r+1}}\left(n-M_{r+1}\right)^{-\alpha-1}\sqrt{M_{r+1}}$$

$$\leq c\left(\alpha\right)n^{\alpha}\sum_{r=0}^{k-2}\frac{M_{r+1}}{M_k^{\alpha+1}}\omega\left(1/M_r,f\right)_p$$

$$\leq c\left(\alpha\right)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p.$$

Analogously,we can prove that

$$I_{13}\leq c\left(\alpha\right)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega\left(1/M_r,f\right)_p. \tag{9}$$

It is easy to show that

$$I_{12}=0. \tag{10}$$

Combining (5)-(10) we receive the proof of Lemma 2.

Lemma 3. *Let $f\in L^p(G_m)$ for some $p\in[1,\infty]$. Then for every $\alpha\in(0,1)$ the following estimations hold*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v\left(u\right)\left[f\left(\cdot\oplus u\right)-f\left(\cdot\right)\right]d\mu\left(u\right)\right\|_p$$

$$\leq c\left(p,\alpha\right)\omega\left(1/M_{k-1},f\right)_p M_k^{\alpha}$$

and

$$\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=M_k}^{n}A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p$$

$$\leq c(p,\alpha)\,\omega(1/M_k,f)_p\,M_k^{\alpha}$$

where $M_k\leq n<M_{k+1}$.

Proof of Lemma 3. We can write

$$II=\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p \tag{11}$$

$$=\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(u)f(\cdot\oplus u)\,d\mu(u)\right\|_p$$

$$\leq\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-S_{M_{k-1}}(\cdot\oplus u,f)\right]d\mu(u)\right\|_p$$

$$+\frac{1}{A_n^{-\alpha}}\left\|\int\limits_{G_m}\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(u)S_{M_{k-1}}(\cdot\oplus u,f)\,d\mu(u)\right\|_p=II_1+II_2.$$

It is evident that

$$II_2=0. \tag{12}$$

Using generalized Minkowski's inequality we have

(13)

$$II_1\leq\frac{1}{A_n^{-\alpha}}\int\limits_{G_m}\left|\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(u)\right|\left\|\left[f(\cdot\oplus u)-S_{M_{k-1}}(\cdot\oplus u,f)\right]\right\|_p d\mu(u)$$

$$\leq c(\alpha)\,n^{\alpha}\omega(1/M_{k-1},f)_p\int\limits_{G_m}\left|\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(u)\right|d\mu(u).$$

Let $t\in I_{A-1}\backslash I_A$, $A\in N$ and $M_k=pM_A+q$, where $0\leq q<M_A$.We have

$$\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v(t)=\sum_{v=M_{k-1}}^{pM_A-1}A_{n-v}^{-\alpha}\psi_v(t) \tag{14}$$

$$+\sum_{v=pM_A}^{M_k-1} A_{n-v}^{-\alpha}\psi_v(t)=\sum_{r=M_{k-1}/M_A}^{p-1}\sum_{v=rM_A}^{(r+1)M_A-1} A_{n-v}^{-\alpha}\psi_v(t)$$

$$+\sum_{v=0}^{q-1} A_{n-v-pM_A}^{-\alpha}\psi_{v+pM_A}(t)=\sum_{r=M_{k-1}/M_A}^{p-1}\sum_{v=0}^{M_A-1} A_{n-v-rM_A}^{-\alpha}\psi_{v+rM_A}(t)$$

$$+\sum_{v=0}^{q-1} A_{n-v-pM_A}^{-\alpha}\psi_{v+pM_A}(t)=\sum_{r=M_{k-1}/M_A}^{p-2}\psi_{rM_A}(t)\sum_{v=0}^{M_A-1} A_{n-v-rM_A}^{-\alpha}\psi_v(t)$$

$$+\psi_{(p-1)M_A}(t)\sum_{v=0}^{M_A-1} A_{n-v-(p-1)M_A}^{-\alpha}\psi_v(t)$$

$$+\frac{1}{A_n^{\alpha}}\psi_{rM_A}(t)\sum_{v=0}^{q-1} A_{q-v}^{-\alpha}\psi_v(t)=A_1+A_2+A_3.$$

Estimate A_1. Since $D_{M_A}(t)=0, t\in I_{A-1}\backslash I_A$, using Abelian transformation,we have

$$|A_1|=\left|\sum_{r=M_{k-1}/M_A}^{p-1}\psi_{rM_A}(t)\sum_{v=0}^{M_A-2} A_{n-v-rM_A}^{-\alpha-1}D_v(t)\right|.$$

Since $|D_k(t)|\le k$, $t\in G_m$, we obtain

$$|A_1|\le c(\alpha)M_A\sum_{r=M_{k-1}/M_A}^{p-2}\sum_{v=0}^{M_A}(n-rM_A-v)^{-\alpha-1} \tag{15}$$

$$\le c(\alpha)M_A(n-(p-1)M_A)^{-\alpha}\le c(\alpha)M_A^{1-\alpha}.$$

For A_2 we have

$$|A_2|\le c(\alpha)\frac{1}{A_n^{\alpha}}\sum_{v=0}^{M_A-1}(n-(p-1)M_A-v)^{-\alpha} \tag{16}$$

$$\le c(\alpha)\sum_{v=0}^{M_A-1}(M_A+q-v)^{-\alpha}\le c(\alpha)M_A^{1-\alpha}.$$

We can write

$$|A_3|\le c(\alpha)\sum_{v=0}^{q-1}(q-v)^{-\alpha} \tag{17}$$

$$\le c(\alpha)q^{1-\alpha}\le c(\alpha)M_A^{1-\alpha}.$$

Combining (14)-(17) we obtain

$$\left|\sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha}\psi_v(u)\right| \leq c(\alpha) M_A^{1-\alpha}, \qquad t \in I_{A-1}\backslash I_A. \tag{18}$$

We can write

(19)

$$\int_{G_m}\left|\sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha}\psi_v(u)\right| d\mu(u) = \sum_{A=1}^{k}\int_{I_{A-1}\backslash I_A}\left|\sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha}\psi_v(u)\right| d\mu(u)$$

$$+\int_{I_k}\left|\sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha}\psi_v(u)\right| d\mu(u) \leq c(\alpha)\sum_{A=1}^{k}\frac{1}{M_A}M_A^{1-\alpha}$$

$$+\frac{c(\alpha)}{M_k}M_k^{1-\alpha} \leq c(\alpha).$$

Combining (18)-(19) we have

$$II_1 \leq c(\alpha)\,\omega(1/M_{k-1}, f)_p\, M_k^{\alpha}. \tag{20}$$

Combining (11)-(20) we conclude that

$$\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p \tag{21}$$

$$\leq c(\alpha)\,\omega(1/M_{k-1}, f)_p\, M_k^{\alpha}.$$

Analogously,we can prove that

$$\frac{1}{A_n^{-\alpha}}\left\|\int_{G_m}\sum_{v=M_k}^{n} A_{n-v}^{-\alpha}\psi_v(u)\left[f(\cdot\oplus u)-f(\cdot)\right]d\mu(u)\right\|_p \tag{22}$$

$$\leq c(\alpha)\,\omega(1/M_k, f)_p\, M_k^{\alpha}.$$

Lemma 3 proved.

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1. We can write

$$\sigma_n^{-\alpha}(f,x)-f(x)=\frac{1}{A_n^{-\alpha}}\int\limits_{G_m}\sum_{v=0}^{n}A_{n-v}^{-\alpha}\psi_v(x)\left[f\left(\cdot\oplus u\right)-f\left(\cdot\right)\right]d\mu\left(u\right)$$

$$=\frac{1}{A_n^{-\alpha}}\int\limits_{G_m}\sum_{v=0}^{M_{k-1}-1}A_{n-v}^{-\alpha}\psi_v\left(x\right)\left[f\left(\cdot\oplus u\right)-f\left(\cdot\right)\right]d\mu\left(u\right)$$

$$+\frac{1}{A_n^{-\alpha}}\int\limits_{G_m}\sum_{v=M_{k-1}}^{M_k-1}A_{n-v}^{-\alpha}\psi_v\left(x\right)\left[f\left(\cdot\oplus u\right)-f\left(\cdot\right)\right]d\mu\left(u\right)$$

$$+\frac{1}{A_n^{-\alpha}}\int\limits_{G_m}\sum_{v=M_k}^{n}A_{n-v}^{-\alpha}\psi_v\left(x\right)\left[f\left(\cdot\oplus u\right)-f\left(\cdot\right)\right]d\mu\left(u\right)=I+II+III$$

Since

$$\left\|\sigma_n^{-\alpha}\left(f,x\right)-f\left(x\right)\right\|_p\leq\left\|I\right\|_p+\left\|II\right\|_p+\left\|III\right\|_p$$

From Lemma 2 and Lemma 3 the proof of theorem is complete.

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