# ON THE CONVERGENCE OF CESÀRO MEANS OF NEGATIVE ORDER OF WALSH-FOURIER SERIES

GVANTSA SHAVARDENIDZE AND MARIAM TOTLADZE

ABSTRACT. In this paper we investigate convergence of Cesàro means of negative order of Walsh-Fourier series of functions of generalized bounded oscilation

Let $r_0(x)$ be a function defined on $R := (-\infty, \infty)$ by

$$r_0(x) = \begin{cases} 1, \text{if } x \in \left[0, \frac{1}{2}\right) \\ -1, \text{if } x \in \left[\frac{1}{2,} 1\right) \end{cases}, \quad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x) \quad n \geq 1 \text{ and } x \in [0, 1).$$

Let $w_0, w_{1...}$ represent the Walsh function, i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + ... + 2^{n_s}$ is a positive integer with $n_1 > n_2 > ... > n_s$ then $w_k(x) = r_{n_1}(x) \times ... \times r_{n_s}(x)$.

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [16].

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, \text{ if } x \in \left[0, \frac{1}{2^n}\right) \\ 0, \text{ if } x \in \left[\frac{1}{2^n}, 1\right) \end{cases}$$

Suppose that $f$ is a Lebesgue integrable function on $[0, 1]$ and 1-periodic. Then its Walsh-Fourier series is defined by

$$\sum_{k=0}^{\infty} \widehat{f}(k) w_k(x),$$

where

$$\widehat{f}(k) = \int_0^1 f(t) w_k(t) \, dt$$

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[0]2010 Mathematics Subject Classification 42C10 .

Key words and phrases: Walsh-Fourier series, Cesáro means, Generalized bounded variation

is called the kth Walsh-Fourier coefficient of function $f$. Denote by $S_n(f,x)$ the nth partial sum of the Walsh-Fourier series of the function $f$, namely

$$S_n(f,x)=\sum_{k=0}^{n-1}\widehat{f}(k)w_k(x).$$

The Cesàro $(C,\alpha)$-means of the Walsh-Fourier series are defined as

$$\sigma_n^\alpha(f,x)=\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^\alpha\widehat{f}(k)\,w_k(x),$$

where

$$A_0^\alpha=1$$

$$A_n^\alpha=\frac{(\alpha+1)\ldots(\alpha+n)}{n!}\quad \alpha\neq-1,-2,\ldots$$

Let $C([0,1])$ denote the space of continuous functions $f$ with period-1. if $f\in C([0,1])$ then the function

$$w(\delta,f)=\sup\left\{\left|f(x')-f(x'')\right|:\left|x'-x''\right|\leq\delta,\quad x',x''\in[0,1]\right\}$$

is called the modulus of continuity of the function $f$.

The modulus of continuity of an arbitrary function $f\in C([0,1])$ has the following properties:

1) $\omega(0)=0$,
2) $\omega(\delta)$ is nondecreasing,
3) $\omega(\delta)$ is continuous on $[0,1]$,
4) $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$ for $0\leq\delta_1\leq\delta_2\leq\delta_1+\delta_2\leq1$.

An arbitrary function $\omega(\delta)$ which is defined on $[0,1]$ and has properties 1)-4) is called a modulus of continuity.If the modulus of continuity $\omega(\delta)$ is given, then $H^\omega$ denotes the class of functions $f\in C([0,1])$ for which

$$\omega(\delta,f)=O(\omega(\delta))\quad\text{as}\quad\delta\to0.$$

$C_w([0,1])$ is the collection of functions $f\in[0,1)\to R$ that are uniformly continuous from the dyadic topology of $[0,1)$ to the usual topology of $R$, or for short: uniformly $W$-continuous.

Let $f$ be defined on $[0,1)$.We shall represent the dyadic modulus of continuity by

$$\dot{\omega}(\delta,f)=\sup_{0\leq h\leq\delta}\sup_x|f(x\oplus h)-f(x)|$$

where $\oplus$ denotes dyadic addition (see [12] or [18]).

The problems of summability of Cesàro means of Walsh-Fourier series were studied in [4] ,[7] ,[10] ,[9] ,[8] ,[16] ,[18] ,[17] .

Tevzadze [19] has studied the uniform convergence of Cesàro means of negative order of Walsh-Fourier series. In particular, in terms of modulus of continuity and variation of function $f\in C_w([0,1])$ he has proved the criterion for the uniform summability by the Cesàro method of negative order of Fourier series with respect to the Walsh system.

In [9] Goginava investigated the problem of estimating the deviation of $f \in L_p$ from its Cesàro means of negative order of Walsh-Fourier series in the $L_p$-metric, $p \in [1, \infty)$. Analogous results for Walsh-Kaczmarz system was proved by Nagy [15] and Gát, Nagy [6].

In his monograph [23, part 1,chap 4] Zhizhiashvili investigated the behavior of Cesàro means of negative order of trigonometric Fourier series in details.

The notion of a function bounded variation was introduced by Jordan [13]. Generalizing this notion Wiener [21] considered the class of function $V_p$. Young [22] introduced the notion of function of bounded $\Phi$-variation. Waterman [20] studied the class of function of bounded $\Lambda$-variation, and Chanturia [3] defined the notion of the modulus of variation of a function. In 1990, Kita and Yoneda [14] introduced the notion of the generalized Wiener's class $BV(p(n) \uparrow p)$. Generalizing the class $BV(p(n) \uparrow p)$ ,Akhobadze [1, 2] considered the classes of function $BV(p(n) \uparrow p, \varphi)$ and $B\Lambda(p(n) \uparrow p, \varphi)$.

**Definition 1.** [11] *Let* $1 \leq p(n) \uparrow p$ *as* $n \to \infty$ *where* $1 \leq p \leq \infty$.*We say that a function belongs to the* $BO(p(n) \uparrow p)$ *class if*

$$O(f; p(n) \uparrow p) := \sup_n \left\{ \sum_{l=0}^{2^n-1} \sup_{t,u \in [l2^{-n}.(l+1)2^{-n})} |f(t) - f(u)|^{p(n)} \right\}^{\frac{1}{p(n)}} < \infty,$$

When $p(n) = p$ for all $n$, $BO(p(n) \uparrow p)$ coincides with the class of $p$-bounded fluctuation $BF_p$ [18].

Estimates of the Fourier coefficients of functions of bounded fluctuation with respect to the Vilenkin system were studied by Gát and Toledo [5].

In [11] Goginava proved that the following statemants are true.

**Theorem G1.** *Let* $f$ *be a function in the class* $BO(p(n) \uparrow \infty)$ *and*

$$\dot{\omega}\left(\frac{1}{2^n}, f\right) = o\left(\frac{1}{p(n+1)\log_2 p(n+1)}\right) \quad as\ n \to \infty.$$

*Then the Walsh-Fourier series of* $f$ *converges uniformly in* $[0, 1]$.

**Theorem G2.** *Let* $p(2n) \leq cp(n)$, $n \in P$ *and* $p(n)\log_2 p(n) = o(n)$ *as* $n \longrightarrow \infty$. *If* $\omega$ *satisfies the condition*

$$\lim_{n \longrightarrow \infty} \sup \omega\left(\frac{1}{n}\right) p([\log_2 n]) \log_2 p([\log_2 n]) = c_0 > 0$$

*then there exists a function in the class* $H^\omega \cap BO(p(n) \uparrow \infty)$ *for which the Walsh-Fourier series diverges at some point.*

Theorem of Tevzadze [19] imply that if $p < \frac{1}{\alpha}$ and $f \in BF_p \cap C_\omega$, then Cesàro mean $\sigma_n^{-\alpha}(f)$ of Walsh-Fourier series uniformly convergence to the function $f$. On the other hand, for $p = \frac{1}{\alpha}$ there exists continuous function $f$, for which $\sigma_n^{-\alpha}(f, 0)$ diverges. On the basis of the above facts the following problems arise naturally:

Let $f \in BO\left(p(n) \uparrow \frac{1}{\alpha}\right)$, $0 < \alpha < 1$.Under what condition on the sequence $\{p(n) : n \geq 1\}$ the uniform convergence of Cesàro $(C, -\alpha)$ means of Walsh-Fourier series of the function f holds.

The following theorem is true.

**Theorem 1.** *Let $f \in C_w([0,1]) \cap BO\left(p(n) \uparrow \frac{1}{\alpha}\right)$, $0 < \alpha < 1$, $2^k \leq n \leq 2^{k+1}$ then*

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_c$$

$$\leq c(\alpha)\left\{\sum_{r=0}^{k} 2^{r-k}\dot{\omega}\left(\frac{1}{2^r}, f\right)_c + \frac{\left(\dot{\omega}\left(\frac{1}{2^k}, f\right)\right)^{1-\alpha p(k)}}{1-\alpha p(k)}\right\}.$$

**Corollary 1.** *Let $f \in C_w([0,1]) \cap BO\left(p(n) \uparrow \frac{1}{\alpha}\right)$, $0 < \alpha < 1$ and*

$$\frac{\left(\dot{\omega}\left(\frac{1}{2^k}, f\right)\right)^{1-\alpha p(k)}}{1-\alpha p(k)} \to 0, \quad \textit{as } k \to \infty,$$

*then*

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_c \to 0.$$

In order to prove Theorem 1 we need the following lemmas proved by Goginava in [9, 8]

**Lemma 1** (Goginava [9])**.** *Let $f \in C_w([0,1])$ . Then for every $\alpha \in (0,1)$ the following estimation holds*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=0}^{2^{k-1}-1} A_{n-\nu}^{-\alpha} w_\nu(u)\left[f(\cdot \oplus u) - f(\cdot)\right] du\right\|_c$$

$$\leq c(p,\alpha)\sum_{r=0}^{k-1} 2^{r-k}\dot{\omega}\left(1/2^r, f\right)_p,$$

*where $2^k \leq n < 2^{k+1}$.*

**Lemma 2** (Goginava [8])**.** *Let $f \in C_w([0,1])$ and $2^k \leq n < 2^{k+1}$. Then for every $\alpha \in (0,1)$ the following estimations holds*

$$\frac{1}{A_n^{-\alpha}}\left|\int_0^1 \sum_{\nu=2^{k-1}}^{2^k-1} A_{n-\nu}^{-\alpha} w_\nu(u)\left[f(\cdot \oplus u) - f(\cdot)\right] du\right|$$

$$\leq \quad c(\alpha)\left(\sum_{j=1}^{2^{k-1}-1} \frac{1}{j^{1-\alpha}}\left|f\left(x \oplus \frac{2j}{2^k}\right) - f\left(x \oplus \frac{2j+1}{2^k}\right)\right|\right),$$

$$\frac{1}{A_n^{-\alpha}}\left|\int_0^1 \sum_{\nu=2^k}^{n} A_{n-\nu}^{-\alpha} w_\nu (u) \left[f(\cdot \oplus u) - f(\cdot)\right] du\right|$$

$$\leq \quad c(\alpha)\left(\sum_{j=1}^{2^k} \frac{1}{j^{1-\alpha}}\left|f\left(x \oplus \frac{2j}{2^{k+1}}\right) - f\left(x \oplus \frac{2j+1}{2^{k+1}}\right)\right|\right).$$

*Proof of Theorem 1.* We can write

$$\sigma_n^{-\alpha}(f,x) - f(x) \tag{1}$$

$$= \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{n} A_{n-\nu}^{-\alpha} w_\nu (x) \left[f(x \oplus u) - f(x)\right] du$$

$$= \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=0}^{2^{k-1}-1} A_{n-\nu}^{-\alpha} w_\nu (x) \left[f(x \oplus u) - f(x)\right] du$$

$$+\frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=2^{k-1}}^{2^k-1} A_{n-\nu}^{-\alpha} w_\nu (x) \left[f(x \oplus u) - f(x)\right] du$$

$$+\frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{\nu=2^k}^{n} A_{n-\nu}^{-\alpha} w_\nu (x) \left[f(x \oplus u) - f(x)\right] du$$

$$= I + II + III.$$

From Lemmas 1 and 2 we have

$$\|I\|_c \leq c(\alpha) \sum_{\nu=0}^{k-1} 2^{r-k} \omega\left(\frac{1}{2^r}, f\right)_c, \tag{2}$$

$$|II| \leq c(\alpha)\left(\sum_{j=1}^{2^{k-1}-1} \frac{1}{j^{1-\alpha}}\left|f\left(x \oplus \frac{2j}{2^k}\right) - f\left(x \oplus \frac{2j+1}{2^k}\right)\right|\right) \tag{3}$$

and

$$|III| \leq c(\alpha)\left(\sum_{j=1}^{2^k-1} \frac{1}{j^{1-\alpha}}\left|f\left(x \oplus \frac{2j}{2^{k+1}}\right) - f\left(x \oplus \frac{2j+1}{2^{k+1}}\right)\right|\right). \tag{4}$$

Using Abel's transformation, we get

$$|III| \leq c(\alpha)\left(\sum_{j=1}^{2^k-2}\left(\frac{1}{j^{1-\alpha}} - \frac{1}{(j+1)^{1-\alpha}}\right)\right. \tag{5}$$

$$\times \sum_{l=1}^{j}\left|f\left(x \oplus \frac{2l}{2^{k+1}}\right) - f\left(x \oplus \frac{2l+1}{2^{k+1}}\right)\right|$$

$$+\frac{1}{\left(2^k-1\right)^{1-\alpha}}\sum_{j=1}^{2^k-1}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|\Bigg)$$
$$=III_1+III_2.$$

Let $\varepsilon_k:=\alpha p_k<1, s_k:=\frac{p(k)}{\varepsilon_k}, \frac{1}{s_k}+\frac{1}{t_k}=1$. Then using Hölder's inequality for $III_2$ we can write

$$III_2=\frac{1}{\left(2^k-1\right)^{1-\alpha}}\sum_{j=1}^{2^k-1}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|^{\varepsilon_k} \tag{6}$$
$$\times\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|^{1-\varepsilon_k}$$
$$\leq\frac{c(\alpha)}{2^{k(1-\alpha)}}\left(\sum_{j=1}^{2^k-1}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|^{p(k)}\right)^{\frac{\varepsilon_k}{p(k)}}$$
$$\times\left(\sum_{j=1}^{2^k-1}\left|f\left(x\oplus\frac{2j}{2^{k+1}}\right)-f\left(x\oplus\frac{2j+1}{2^{k+1}}\right)\right|^{(1-\varepsilon_k)t_k}\right)^{\frac{1}{t_k}}$$
$$\leq\frac{c(\alpha)}{2^{k(1-\alpha)}}\left(BO\left(f,p(k)\uparrow\frac{1}{\alpha}\right)\right)^{\varepsilon_k}\left(\dot{\omega}\left(f,\frac{1}{2^k}\right)\right)^{1-\varepsilon_k}2^{\frac{k}{t_k}}$$
$$\leq c(\alpha)\left(BO\left(f,p(k)\uparrow\frac{1}{\alpha}\right)\right)^{\varepsilon_k}\left(\dot{\omega}\left(f,\frac{1}{2^k}\right)\right)^{1-\varepsilon_k}2^{k\left(\alpha-\frac{1}{s_k}\right)}$$
$$=c(\alpha)\left(BO\left(f,p(k)\uparrow\frac{1}{\alpha}\right)\right)^{\varepsilon_k}\left(\dot{\omega}\left(f,\frac{1}{2^k}\right)\right)^{1-\varepsilon_k}2^{k\left(\alpha-\frac{\varepsilon_k}{p(k)}\right)}$$
$$=c(\alpha)\left(BO\left(f,p(k)\uparrow\frac{1}{\alpha}\right)\right)^{\varepsilon_k}\left(\dot{\omega}\left(f,\frac{1}{2^k}\right)\right)^{1-\alpha p(k)}\to 0$$

as $k\to\infty$.

Fix $m_0(k)$ and define it later

$$III_1\leq c(\alpha)\sum_{j=1}^{m_0(k)}\frac{1}{j^{2-\alpha}}\sum_{l=1}^{j}\left|f\left(x\oplus\frac{2l}{2^{k+1}}\right)-f\left(x\oplus\frac{2l+1}{2^{k+1}}\right)\right| \tag{7}$$
$$+\sum_{j=m_0(k)+1}^{2^k-1}\frac{1}{j^{2-\alpha}}\sum_{l=1}^{j}\left|f\left(x\oplus\frac{2l}{2^{k+1}}\right)-f\left(x\oplus\frac{2l+1}{2^{k+1}}\right)\right|$$
$$\leq c(\alpha)\left\{\sum_{j=1}^{m_0(k)}\frac{1}{j^{2-\alpha}}j\dot{\omega}\left(\frac{1}{2^k},f\right)\right.$$
$$\left.+\sum_{j=m_0(k)+1}^{2^k-1}\frac{1}{j^{1+1/p(k)-\alpha}}\left(\sum_{l=1}^{j}\left|f\left(x\oplus\frac{2l}{2^{k+1}}\right)-f\left(x\oplus\frac{2l+1}{2^{k+1}}\right)\right|^{p(k)}\right)^{\frac{1}{p(k)}}\right\}$$

$$\leq c(\alpha)\left\{(m_0(k))^{\alpha}\dot{\omega}\left(\frac{1}{2^k},f\right)+\frac{m_0(k)^{\alpha-\frac{1}{p(k)}}}{\frac{1}{p(k)}-\alpha}BO\left(f,p(k)\uparrow\frac{1}{\alpha}\right)\right\}.$$

Set

$$m_0(k)=\left(\frac{1}{\dot{\omega}\left(\frac{1}{2^k},f\right)}\right)^{p(k)}.$$

Then we have

$$III_1\leq c(\alpha)\left\{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}+\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{\frac{1}{p(k)}-\alpha}\right\}$$

$$\leq c(\alpha)\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{1-\alpha p(k)}. \tag{8}$$

Combining (5)-(8) we have

$$|III|\leq c(\alpha)\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{1-\alpha p(k)}. \tag{9}$$

Analogously we can proved that

$$|II|\leq c(\alpha)\frac{\dot{\omega}\left(\frac{1}{2^k},f\right)^{1-\alpha p(k)}}{1-\alpha p(k)} \tag{10}$$

Combining (1), (2) , (9) and (10) we complete the proof of Theorem 1. □

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G.Shavardenidze, Department of Mathematics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
*E-mail address*: shavardenidzegvantsa@gmail.com

M.Totladze, Department of Mathematics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
*E-mail address*: totladzemariam@gmail.com